1. Find the total area between the curves $y = x\sqrt{1-x^2}$ and $y = 0$.

Solution

First, notice the curves intersect at $x = 0, 1, -1$, so we are looking for the area between the curves from $x = -1$ to $x = 0$ and then from $x = 0$ to $x = -1$. Notice by graphing or otherwise that the area in each of those intervals is the same. ie.

$$A = \int_{-1}^{0} -x\sqrt{1-x^2}dx + \int_{0}^{1} x\sqrt{1-x^2}dx = 2\int_{0}^{1} x\sqrt{1-x^2}dx$$

Letting $u = 1 - x^2$ we have the following

$$\begin{aligned} A &= -\int_{1}^{0} u^{\frac{1}{2}}du \\ &= \int_{0}^{1} u^{\frac{1}{2}}du \\ &= \frac{2}{3} \end{aligned}$$

2. Sketch and find the area between the curves $y = \sqrt{x+2}$, $y = \frac{x+2}{3}$

Solution

Notice that the curves intersect at $x = -2$ and $x = 7$, and during that interval $\sqrt{x+2} > \frac{x+2}{3}$. Thus, the area between the curves is give by the following:

$$A = \int_{-2}^{7} \sqrt{x+2} - \frac{x+2}{3}dx \qquad = \int_{0}^{9} \sqrt{u} - \frac{u}{3}dx$$

$$= \frac{2u^{3/2}}{3} - \frac{u^2}{6}\Big]_0^9$$

$$= \frac{9}{2}$$

3. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x-2}$, $y = 0$ and $x = 6$, about the $x - axis$.

Solution

When $y = 0$ then $x = 0$ so we are looking for $\int_0^6 A(x)dx$ where $A(x)$ is the area of a cross section. Each cross section is a disk of radius $\sqrt{x-2}$, so $A(x) = \pi(x-2)$.

$$\begin{aligned} V &= \int_0^6 \pi(x-2)dx \\ &= \left[\frac{\pi x^2}{2} - 2\pi x\right]_0^6 \\ &= 6\pi \end{aligned}$$

4. Find the volume of the solid obtained by rotating the region bounded by $x = y^2$ and $x = 3y$ about the y-axis.

Solution

The curves intersect when $y = 0$ and $y = 3$, so we're looking for $\int_0^3 A(y)dy$ where $A(y)$ is the area of the cross section for each y value. Each cross section is a washer with inner radius y^3 and outer radius $3y$, so $A(y) = \pi(9y^2 - y^4)$

$$\begin{aligned} \int_0^3 A(y)dy &= \int_0^3 \pi(9y^2 - y^4)dy \\ &= \pi \int_0^3 9y^2 - y^4 dy \\ &= \pi \left[3y^3 - \frac{y^5}{5}\right]_0^3 \\ &= \frac{162\pi}{5} \end{aligned}$$

5. Find the volume of the solid obtained by rotating the region bounded by the curve $y = x^3$ and the lines $y = 1$ and $x = 2$ about $y = 3$. (Hint: Draw the region and a typical disk or washer).

Solution

Ignore this question.

6. Find the volume of the solid obtained by rotating the region bounded by the curves $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq \frac{\pi}{4}$ about the x-axis. (Hint: Draw the region and a typical disk or washer)

Solution

A cross section perpendicular to the x-axis is a washer of inner radius $\sin x$ and outer radius $\cos x$ and thus $A(x) = \pi(\cos^2 x - \sin^2 x)$. The volume of the solid is as follows:
(Note we will use double angle formula)

$$\begin{aligned} V &= \int_0^{\frac{\pi}{4}} \pi(\cos^2 x - \sin^2 x)dx \\ &= \int_0^{\frac{\pi}{4}} \pi \cos(2x)dx \\ &= \pi \int_0^{\frac{\pi}{4}} \cos(2x)dx \end{aligned}$$

Let $u = 2x$. Then $du = 2dx$ and when $x = 0$, $u = 0$ and when $x = \frac{\pi}{4}$, $u = \frac{\pi}{2}$.

$$\begin{aligned} V &= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \cos(u)du \\ &= \frac{\pi}{2} \sin(u)\Big]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2}\left(\sin(\frac{\pi}{2}) - \sin(0)\right) \\ &= \frac{\pi}{2} \end{aligned}$$

7. Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. (Try using cylindrical shells!)

Solution

$$\begin{aligned} V &= \int_0^1 \pi x dx \\ &= \frac{\pi x^2}{2}\Big]_0^1 \\ &= \frac{\pi}{2} \end{aligned}$$