1. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

Solution

The volume of the pile of gravel is given by $V = \frac{\pi}{3}r^2h$ where r is the radius of the base of the pile, and h is the height of the pile. Since the diameter of the base of the pile is the same as the height, and the radius is half the diameter then $r = \frac{h}{2}$. This means that

$$V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

We know that the volume is increasing at a rate of 30 ft^3/min so $\frac{dV}{dt} = 30$. Differentiating with respect to t, and then letting $h = 10$ we get the following:

$$\begin{aligned}
\frac{dV}{dt} &= \frac{\pi}{12}3h^2 \cdot \frac{dh}{dt} \\
30 &= \frac{\pi}{4}h^2\frac{dh}{dt} \\
30 &= \frac{100\pi}{4}\frac{dh}{dt} \\
30 &= 25\pi\frac{dh}{dt} \\
\frac{30}{25\pi} &= \frac{dh}{dt} \\
\frac{6}{5\pi} &= \frac{dh}{dt}
\end{aligned}$$

So when the pile is 10 ft tall then the height of the pile is increasing at a rate of $\frac{6}{5\pi}$ ft/min

2. Find $\frac{dy}{dx}$ if $y\cos\left(\frac{1}{y}\right) = 5 - 2xy$

Solution

$$\begin{aligned}
\frac{d}{dx}\left(y\cos\left(\frac{1}{y}\right)\right) &= \frac{d}{dx}(5-2xy)\\
\frac{dy}{dx}\cos\left(\frac{1}{y}\right) + y\left(-\sin\left(\frac{1}{y}\right)\cdot\left(-y^{-2}\right)\cdot\frac{dy}{dx}\right) &= -2x\frac{dy}{dx} - 2y\\
\frac{dy}{dx}\cos\left(\frac{1}{y}\right) + \frac{1}{y}\sin\left(\frac{1}{y}\right)\frac{dy}{dx} + 2x\frac{dy}{dx} &= -2y\\
\frac{dy}{dx} &= \frac{-2y}{\cos(\frac{1}{y}) + \frac{1}{y}\sin(\frac{1}{y}) + 2x}\\
\frac{dy}{dx} &= \frac{-2y^2}{y\cos(\frac{1}{y}) + \sin(\frac{1}{y}) + 2xy}
\end{aligned}$$

3. A child flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from them at a rate of 25 ft/sec. How fast must they let out the string when the kite is 500 ft away from them horizontally.

Solution

Let h represent the horizontal distance, and s represent the length of the string. Using Pythagorean theorem, we know that $s = \sqrt{300^2 + h^2}$. We know $\frac{dh}{dt} = 25$, and we are interested in $\frac{ds}{dt}$ when $h = 500$.

$$\begin{aligned}
\frac{d}{dt}s &= \frac{d}{dt}\sqrt{300^2 + h^2}\\
\frac{ds}{dt} &= \frac{1}{2\sqrt{300^2 + h^2}}\cdot 2h\cdot\frac{dh}{dt}\\
\frac{ds}{dt} &= \frac{1}{2\sqrt{300^2 + 500^2}}\cdot 2(500)(25)\\
&= \frac{1}{2\sqrt{300^2 + 500^2}}\cdot 2(500)(25)\\
&= \frac{250}{\sqrt{34}}
\end{aligned}$$

When the kite is 500 ft away from them they should let the string out at a rate of $\frac{250}{\sqrt{34}}$ ft/sec.

4. (i) Find the critical points of the following functions and then (ii) determine the local minimum and maximum values and where they occur.

(a) $f(x) = x^{2/3} - 3x$

Solution

$$f'(x) = \frac{2}{3}x^{-1/3} - 3$$

Notice $f'(x)$ is undefined at $x = 0$, so 0 is a critical number.

$$\begin{aligned} f'(x) &= \frac{2}{3}x^{-1/3} - 3 \\ 0 &= \frac{2}{3}x^{-1/3} - 3 \\ 3 &= \frac{2}{3}x^{-1/3} \\ \frac{9}{2} &= x^{-1/3} \\ \frac{9^3}{2^3} &= x^{-1} \\ \frac{8}{9^3} &= x \\ \frac{8}{279} &= x \end{aligned}$$

So the critical numbers are at 0 and $\frac{8}{729}$. By testing values in each interval, we see that f is increasing on $(-\infty, 0), (0, \frac{8}{729})$, and decreasing on $(\frac{8}{729}, \infty)$.

(b) $f(x) = \frac{1}{\sqrt{3x^2-1}}$

Solution

$$\begin{aligned} f(x) &= (3x^2 - 1)^{-\frac{1}{2}} \\ f'(x) &= -\frac{1}{2}(3x^2 - 1)^{-\frac{3}{2}} \cdot 6 \\ &= \frac{-3x}{(3x^2 - 1)^{\frac{3}{2}}} \end{aligned}$$

Notice the domain of f is the same as the domain of f', which is $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$. $f'(x) = 0$ when $x = 0$, but since 0 is not in the domain, then we have no critical points, and thus no local extrema.

(c) $f(x) = x^3 - 24$

Solution

$f'(x) = 3x^2$, so the only critical point is $x = 0$. However, since $f'(x)$ is always positive, so there are no local extrema.

5. Find antiderivatives of the following functions

   (a) $f(x) = 5x^3 - 2x^4 - 3$

   Solution

   $F(x) = \frac{5}{4}x^4 - \frac{2}{5}x^5 - 3x$

   (b) $f(x) = \cos(x) + 3\sin(x)$

   Solution

   $F(x) = \sin(x) - 3\cos(x)$

   (c) $f(x) = \frac{1}{\sqrt[4]{x}}$

   Solution

   $F(x) = \frac{x^{\frac{3}{4}}}{\frac{3}{4}} = \frac{4x^{\frac{3}{4}}}{3}$

6. Find the absolute minimum and absolute maximum values of $f(x)$ on the given interval

   (a) $f(x) = -\frac{2}{x^2+4}$ on $[0, 5]$

   Solution

   $$f'(x) = \frac{4x}{(x^2+4)^2}$$

   Notice the only critical number of f is $x = 0$.
   $f(0) = -\frac{1}{2}$ and $f(5) = \frac{-2}{29}$ so f has an absolute minimum of $-\frac{1}{2}$ at $x = 0$, and f has an absolute maximum of $\frac{-2}{29}$ at $x = 5$.

   (b) $f(x) = -x^3 - 6x^2 - 9x + 3$ on $[-3, -1]$

   Solution

   $$f'(x) = -3x^2 - 12x - 9 = -3((x+3)(x+1)$$

   Notice the only critical numbers of f are -3 and -1, which happen to be the end points.
   $f(-1) = 7$ and $f(-3) = 3$ so f has an absolute maximum of 7 at $x = -1$ and an absolute minimum of 3 at $x = -3$.

7. A particle moves along an axis modeled by the position function $s(t) = t^3 - 7t^2 + 16t - 10$, where s is inches, and t is time in seconds.

   (a) When is the particle moving forwards? When is it moving backwards?

   Solution

   $$v(t) = s'(t) = (3t-8)(t-2)$$

   So the function has critical points at $\frac{8}{3}$ and 2. By testing points in each interval, we see that $v(t)$ is positive in the interval $(0,2)$ and $(\frac{8}{3},\infty)$ and negative in $(2,\frac{8}{3})$. Hence the partical is moving forward from 0 to 2 seconds and from $\frac{8}{3}$ seconds, and moving backwards from 2 to $\frac{8}{3}$ seconds.

   (b) What is the position of the particle when it has velocity of 8 ft/sec?

   Solution

   $$\begin{aligned} v(t) &= 8 \\ 3t^2 - 14t + 16 &= 8 \\ 3t^2 - 14t - 8 &= 0 \\ (3t-2)(t-4) &= 0 \\ t &= \{\frac{2}{3}, 4\} \end{aligned}$$

   $s(\frac{2}{3}) = -\frac{58}{27}$
   $s(4) = 26$

8. Consider the function $k(x) = \frac{4}{\sqrt{x}}$. (i) Find dy when $y = k(x)$ and (ii)evaluate dy when $dx = 0.01$ and $x = 1$. (iii)Compare dy to Δy

   Solution

   $$dy = -\frac{2}{x^{\frac{3}{2}}}dx$$

   When $x = 1$ and $dx = 0.01$ then $dy = -0.02$
   $\Delta y = k(1.01) - k(1) = \frac{4}{\sqrt{1.01}} - 4 \equiv -.01985$

9. Evaluate the following limits.

   (a) $\lim\limits_{x\to\infty} \frac{4x^2-3x}{2x-1}$

   Solution

(b) $\lim_{x \to \infty} \frac{2x+5}{\sqrt{8x^2-5x+1}}$

Solution

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x+5}{\sqrt{8x^2-5x+1}} &= \lim_{x\to\infty} \frac{2x+5}{\sqrt{8x^2-5x+1}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{2+\frac{5}{x}}{\frac{\sqrt{8x^2-5x+1}}{x}} \\
&= \lim_{x\to\infty} \frac{2+\frac{5}{x}}{\sqrt{\frac{8x^2-5x+1}{x^2}}} \\
&= \frac{\lim\limits_{x\to\infty} 2+\frac{5}{x}}{\sqrt{\lim\limits_{x\to\infty} 8-\frac{5}{x}+\frac{1}{x}^2}} \\
&= \frac{2}{\sqrt{8}} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

10. Find the vertical and horizontal asymptotes of the following function.

$$f(x) = \frac{4x+1}{x^2-3x+2}$$

Solution

Vertical Asymptotes: $x = 2$, $x = 1$
Horizontal Asymptote: $y = 0$

11. Sketch the graph of the curve $y = \frac{x^3}{x^3+1}$

12. (i)Explain how we know that the given equation must have EXACTLY ONE root in the interval $[0, 1]$. Then, (ii) use Newtons method to approximate the root (you only need to find x_2 for a reasonable guess for x_1.

$$x^4 - 12x^3 + 6 = 0$$

Solution

Since f is continuous on $[0, 1]$ and $f(0) > 0$ and $f(1) < 0$ then by Intermediate Value theorem there must be a root in $[0, 1]$. If there were 2 roots, then by Rolles theorem (notice f is continuous and differentiable on the interval), then there would be some c in $(0, 1)$ such that $f'(c) = 0$. However, $f'(x) = 0$ only $x = 9$ or $x = 0$, so there must be exactly one root. Since $f'(0) = 0$, then the reasonable estimate for x_1 would be $x_1 = 1$. In that case $x_2 = 1 - \frac{5}{32} = \frac{27}{32}$.