1. Sketch the graph of a function on $[-3,3]$ that has an absolute maximum but no local maximum.

Answers may vary. It should have a maximum value as an end point and either never be increasing or never be decreasing.

2. Sketch the graph of a function on $[-3,3]$ that has an local minimum but no absolute minimum .

Answers may vary. It should have a discontinuity, and the function should be sometimes increasing and sometimes decreasing.

3. Find the critical numbers of the following functions.

(a) $f(x) = \sqrt{x}(x^2 - 3x)$

Solution

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}}(x^2 - 3x) + \sqrt{x}(2x-3) \\ &= \frac{x^2 - 3x}{2\sqrt{x}} + \frac{2x(2x-3)}{2\sqrt{x}} \\ &= \frac{5x^2 - 9x}{2\sqrt{x}} \end{aligned}$$

Notice that f is undefined at $x = 0$, so 0 is a critical point. Furthermore, $f'(\frac{9}{5}) = 0$. Thus our critical numbers are $x = \{0, \frac{9}{5}\}$.

(b) $f(x) = 2x^3 - 3x^2 - 36x$

Solution

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 36 \\ &= 6(x^2 - x - 6) \\ &= 6(x-3)(x+2) \end{aligned}$$

So $f'(x) = 0$ when $x = 3$ and $x = -2$. Since $f'(x)$ is defined everywhere then our only critical numbers are $\{-2, 3\}$.

(c) $f(\theta) = 2\cos\theta + \sin^2\theta$

Solution

$$\begin{aligned} f'(\theta) &= -2\sin\theta + 2\sin\theta \cdot \cos\theta \\ &= (2\sin\theta)(-1+\cos\theta) \end{aligned}$$

$f'(\theta) = 0$ when $\sin\theta = 0$ or $\cos\theta = 1$.
$\sin\theta = 0$ when $\theta = \pi n$ for any integer n.
$\cos\theta = 1$ when $\theta = 2\pi n$ for any integer n..
The function is differentiable everywhere, hence the critical numbers of f are $\theta = \pi n$ where n is an integer.

4. Find the absolute maximum and absolute minimum values of f on the given interval.

(a) $f(t) = (t^2-4)^3$, $[-2,3]$

Solution

We need to evaluate f at the critical numbers in the interval and end points to determine the global extrema. **Critical numbers:** Notice f is differentiable everywhere so we only need to consider values where the derivative is 0.

$$\begin{aligned} f'(t) &= 0 \\ 0 &= 3(t^2-4)^2 \cdot (2t) \\ 0 &= 6t(t^2-4)^2 \\ t &= \{0, 2, -2\} \end{aligned}$$

$$f(-2) = 0 \quad f(0) = -64 \quad f(2) = 0 \quad f(3) = 125$$

Absolute Maximum of 125 at $t = 3$
Absolute Minimum of -64 at $t = -2$

(b) $f(x) = \frac{x}{x^2-x+1}$ [-2,2]

Solution

We need to evaluate f at the critical numbers in the interval and end points to determine the global extrema. **Critical numbers:** Notice f is differentiable everywhere so we only need to consider values where the derivative is 0.

$$\begin{aligned}
0 &= f'(x) \\
0 &= \frac{(x^2-x+1)^2 - x(2x-1)}{(x^2-x+1)^2} \\
0 &= \frac{-x^2+1}{(x^2-x+1)^2} \\
0 &= \frac{(1-x)(1+x)}{(x^2-x+1)^2} \\
x &= \{-1, 1\}
\end{aligned}$$

$$f(-1) = -\frac{1}{3} \quad f(1) = 1 \quad f(-2) = -\frac{2}{7} \quad f(2) = \frac{2}{3}$$

Absolute Maximum of 1 at $x = 1$
Absolute Minimum of $-\frac{1}{3}$ at $x = -1$

5. Verify that the function satisfies the 3 hypotheses of Rolle's theorem on the given interval. Then, find all numbers c that satisfy the conclusion of Rolle's Theorem.

$$f(x) = \sin(x/2),\ [-\pi/2, 3\pi/2]$$

Solution

First, $\sin(x)$ is continuous and differentiable everywhere. Furthermore, $f(-\pi/2) = -\frac{\sqrt{2}}{2} = f(3\pi/2)$, so f satisfies the the hypotheses of Rolles theorem, therefore, there exists a c in the interval such that $f'(c) = 0$.

$$\begin{aligned} f'(c) &= 00 \qquad\qquad = \cos(c/2)\cdot\frac{1}{2} \\ 0 &= \frac{\cos(c/3)}{2} \\ \frac{c}{2} &= \frac{\pi}{2} + n\pi \\ c &= \pi + n\pi \\ c &= \pi \qquad\qquad \text{(since it must be in the interval)} \end{aligned}$$

6. Very that the function satisfies the hypothesis of the Mean Value Theorem on the given interval, and then fund all numbers c that satisfy the conclusion of the Mean Value Theorem.

$$f(x) = 3x^2 - 4x + 1,\ [0, 2]$$

Solution

Since f is a polynomial then it is continuous and differentiable everywhere, thus by the MVT we have that $f'(c) = \frac{f(2)-f(0}{2-0}$ for some c in the interval $[0, 2]$.

$$\begin{aligned} f'(c) &= \frac{5-1}{2}\ 6c - 4 \qquad\qquad = \frac{4}{2} \\ c &= 1 \end{aligned}$$

7. Show that a polynomial of degree 3 has at most 3 real roots.

Solution

Suppose f is a degree 3 polynomial and has 4 real roots. We wish to show that there would be a contradiction. Let the roots be a_1, a_2, a_3, a_4. Since f is polynomial, it is differentiable and continuous everywhere. We can apply Rolles theorem to the intervals $[a_1, a_2], [a_2, a_3], [a_3, a_4]$, giving us 3 distinct points at which the derivative is 0. However, the derivative of a cubic polynomial is a degree 2 polynomial, which has only 2 roots, giving us a contradiction. Hence, a degree 3 polynomial can have at most 3 real roots.

8. Suppose that $f'(x) \leq 2$ for all x in $[1, 5]$. If $f(5) = 10$ whats the largest $f(1)$ could be?

Solution

Since we say that $f'(x) \leq 2$ for all x, then f must be differentiable everywhere, hence continuous everywhere. By MVT there must exist some c in $[1, 5]$ such that

$$f'(c) = \frac{f(5) - f(1)}{5 - (1)} = \frac{f(5) - 10}{4}$$

So $f(5) = 4f'(c) + 10 \leq 4(2) + 10 = 18$. So $f(5)$ can be no larger than 18.

9. Bonus: A number a is called a fixed point of a function f is $f(a) = a$. Prove that if $f'(x) \neq 1$, and $f'(x)$ exists for all for all real numbers x then f has at most one fixed points.

Solution

Proof. Suppose that $f(a) = a$ and $f(b) = b$ for $a < b$. Since f is differentiable everywhere then we can use the mean value theorem to say that $f'(c) = \frac{f(b)-f(a)}{b-a} = 1$ for some c in $[a, b]$. However, this is a contradiction since $f'(x) \neq 1$ so it must be that f cannot have a fixed point. □