1. Find the following derivatives for the function $f(x) = (x^2 - 2x)^3$.

   (a) $f'(x)$

   Solution

   Using chain rule, we get $f'(x) = 3(x^2 - 2x)^2(2x - 2)$.

   (b) $f''(x)$

   Solution

   Using product rule and then chain rule,

   $$f'(x) = 3(x^2 - 2x)^2 \cdot 2 + 3(2)(x^2 - 2x)(2x - 2)(2x - 2)$$

   Simplifying, we have

   $$f'(x) = 6(x^2 - 2x)^2 + 6(x^2 - 2x)(2x - 2)^2$$

   (c) $f'(0)$

   Solution

   $$f'(0) = 3(0)^2(-2) = 0$$

   (d) $f''(0)$

   Solution

   $$f''(0) = 3 \cdot 0^2 + 6(0)(-2)^2 = 0$$

2. Find the following derivatives for the function $f(x) = 3\sin(x)$

   (a) $f'(x)$

   Solution

   $$f'(x) = 3\cos(x)$$

   (b) $f''(x)$

   Solution

   $$f''(x) = -3sin(x)$$

(c) $f^{(11)}(x)$

Solution

Notice that $f^{(11)}(x) = f^{(3)}(x) = -3\cos(x)$

(d) $f^{(20)}(x)$

Solution

$$f^{(20)}(x) = f(x) = 3\sin(x)$$

3. Find the derivatives of the following functions.

(a) $f(x) = \sqrt{\cos(2x) + \sin(2x)}$

Solution

Let's write f as the composition of 3 functions.

$$f(x) = g(h(k(x)))$$

$$g(x) = \sqrt{x}$$
$$h(x) = \cos(x) + \sin(x)$$
$$k(x) = 2x$$

By chain rule, we know that

$$\begin{aligned} f'(x) &= g'(h(k(x))) \cdot h'(k(x)) \cdot k'(x) \\ &= \frac{1}{2\sqrt{\cos(2x) + \sin(2x)}} \cdot (-\sin(2x) + \cos(2x)) \cdot 2 \\ &= \frac{\cos(2x) - \sin(2x)}{\sqrt{\cos(2x) + \sin(2x)}} \end{aligned}$$

(b) $G(x) = \left(\frac{3x-1}{2x^2-1}\right)^3$

Solution

$G(x) = (f \circ g)(x)$ where $f(x) = x^3$ and $g(x) = \frac{3x-1}{2x^2-1}$. We can use chain rule to find $G'(x)$.

$$\begin{aligned} G'(x) &= 3\left(\frac{3x-1}{2x^2-1}\right)^2 \cdot \left(\frac{(2x^2-1)(3) - (3x-1)(4x)}{(2x^2-1)^2}\right) \\ &= 3\left(\frac{3x-1}{2x^2-1}\right)^2 \cdot \left(\frac{-6x^2 - 3 + 4x}{(2x^2-1)^2}\right) \end{aligned}$$

(c) $y = x^{-3}\sec(x)$

Solution

$$\frac{dy}{dx} = -3x^{-4}\sec(x) + x^{-3}\sec(x)\tan x$$

4. For two functions f, g we have the following values. Find the derivatives

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-3	1	2	7	0
2	5	2	-6	-1
5	$\frac{1}{2}$	7	-3	4

(a) $(f \circ g)'(5)$

Solution

$$\begin{aligned}(f \circ g)'(5) &= f'(g(5)) \cdot g'(5) \\ &= f'(-3) \cdot 4 \\ &= 2 \cdot 4 \\ &= 8\end{aligned}$$

(b) $(fg)'(5)$

Solution

$$\begin{aligned}(fg)'(5) &= f'(5)g(5) + f(5)g'(5) \\ &= 6(-3) + 4(\frac{1}{2}) \\ &= -21 + 2 \\ &= -19\end{aligned}$$

(c) $\left(\frac{f}{g}\right)'(5)$

Solution

$$\left(\frac{f}{g}\right)'(5) = \frac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2}$$
$$= \frac{(03)97) - (\frac{1}{2})(4)}{(-3)^2}$$
$$= \frac{-21 - 2}{9}$$
$$= \frac{-23}{9}$$

5. Find the equation of the tangent line to the curve $y = (1 + x^2)\sin(2x)$ at the point $(0, 1)$

Solution

$$\frac{dy}{dx} = (2x)\sin(2x) + (1 + x^2)\cos(2x) \cdot 2$$
$$= (2x)\sin(2x) + 2(1 + x^2)\cos(2x)$$

At $(0, 1)$ $\frac{dy}{dx} = 0 \cdot 1 + 2 \cdot 1 \cdot 1 = 2$, and this is the slope of the tangent line. Notice that $(0, 1)$ is the y-intercept, so the equation of the tangent line to the curve y at $(0, 1)$ is

$$y = 2x + 1$$

6. Find the following limit.

$$\lim_{x\to 3} \frac{\sin(x-3)}{x^2-4x+3}$$

Solution

First, notice that $\lim_{x\to 3} \frac{\sin(x-3)}{x^2-4x+3} = \lim_{x\to 3} \frac{\sin(x-3)}{(x-3)(x-1)}$. We wish to use the fact that $\lim \_x \to 0 frac \sin(x)x = 1$. Let $w = x-3$. Then $x-1 = w+2$ and as $x \to 3$, $w \to 0$.

$$\begin{aligned}
\lim_{x\to 3} \frac{\sin(x-3)}{(x-3)(x-1)} &= \lim_{w\to 0} \frac{\sin(w)}{w(w+2)} \\
&= \lim_{w\to 0} \frac{\sin(w)}{w} \cdot 1w+2 \\
&= \lim_{w\to 0} \frac{\sin(w)}{w} \cdot \lim_{w\to 0} 1w+2 \\
&= 2 \lim_{w\to 0} \frac{\sin(w)}{w} \\
&= 2 \cdot 1 = 2
\end{aligned}$$

7. Find $\frac{dy}{dx}$ for the curve $y^2 + (x-2)^2 = 20$ at the point $(4,-4)$

Solution

$$\begin{aligned}
\frac{d}{dx}[y^2 + (x-2)^2] &= \frac{d}{dx}20 \\
2y\frac{dy}{dx} + 2(x-2)(1) &= 0 \\
2y\frac{dy}{dx} &= -2(x-2) \\
\frac{dy}{dx} &= \frac{-2(x-2)}{2y} \\
&= \frac{-(x-2)}{y} \\
&= \frac{2-x}{y}
\end{aligned}$$

At point $(4,-4)$ $\frac{dy}{dx} = \frac{2-4}{-4} = \frac{1}{2}$.