Goal: complete the following problems without using notes or calculators.

1. Calculate the following limits.

   (a) $\lim_{x \to 0} \frac{\sqrt{3x+4}-2}{x}$

Solution

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{3x+4}-2}{x} &= \lim_{x \to 0} \frac{\sqrt{3x+4}-2}{x} \cdot \frac{\sqrt{3x+4}+2}{\sqrt{3x+4}+2} \\
&= \lim_{x \to 0} \frac{3x+4-4}{x(\sqrt{3x+4}+2)} \\
&= \lim_{x \to 0} \frac{3x}{x(\sqrt{3x+4}+2)} \\
&= \lim_{x \to 0} \frac{3}{(\sqrt{3x+4}+2)} \\
&= \frac{3}{(\sqrt{3(0)+4}+2)} \\
&= \frac{3}{4}
\end{aligned}$$

   (b) $\lim_{x \to 3} \left(\frac{\sin(3-x)}{x-3} + \frac{3x^2+x-2}{x+1} \right)$

Solution

$$\begin{aligned}
\lim_{x \to 3} \left(\frac{\sin(3-x)}{x-3} + \frac{3x^2+x-2}{x+1} \right) &= \lim_{x \to 3} \frac{\sin(3-x)}{x-3} + \lim_{x \to 3} \frac{3x^2+x-2}{x+1} \\
&= \left(\lim_{w \to 0} \frac{\sin(w)}{-w} \right) + \frac{3(3)^2+3-2}{3+1} \\
&= -\left(\lim_{w \to 0} \frac{\sin(w)}{w} \right) + \frac{28}{4} \\
&= -1 + 7 \\
&= 6
\end{aligned}$$

(c) $\lim_{x\to 0} 3x^2 \cos(\frac{2}{x})$

Solution

$$-1 \leq \cos(\frac{2}{x}) \leq 1$$

$$-x^2 \leq x^2 \cos(\frac{2}{x}) \leq x^2$$

$$-3x^2 \leq 3x^2 \cos(\frac{2}{x}) \leq 3x^2$$

Since $\lim_{x\to 0} -3x^2 = 0 = \lim_{x\to 0} 3x^2$, then by the Squeeze theorem, since $\lim_{x\to 0} 3x^2 \cos(\frac{2}{x}) = 0$

(d) $\lim_{x\to 3^-} \frac{|x-3|}{x-3}$

Solution

As x is approaching 3 from the left, then $x - 3 < 0$ so $|x - 3| = -(x - 3)$.

$$\begin{aligned}\lim_{x\to 3^-} \frac{|x-3|}{x-3} &= \lim_{x\to 3^-} \frac{-(x-3)}{x-3} \\ &= \lim_{x\to 3^-} -1 \\ &= -1\end{aligned}$$

(e) $\lim_{x\to 0} \frac{\sin(3x)}{4x}$

Solution

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(3x)}{4x} &= \lim_{x\to 0} \frac{\sin(3x)}{4x} \cdot \frac{\frac{3}{4}}{\frac{3}{4}} \\ &= \frac{3}{4} \lim_{x\to 0} \frac{\sin(3x)}{3x} \\ &= \frac{3}{4} \cdot 1 = \frac{3}{4}\end{aligned}$$

(f) $\lim\limits_{x\to 0} \frac{\cos(2x)-1}{3x}$

Solution

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos(2x)-1}{3x} &= \lim_{x\to 0} \frac{\cos(2x)-1}{3x} \cdot \frac{\frac{2}{3}}{\frac{2}{3}} \\
&= \frac{2}{3} \lim_{x\to 0} \frac{\cos(2x)-1}{2x} \\
&= \frac{2}{3} \cdot 0 \\
&= 0
\end{aligned}$$

2. Use the definition of the derivative to find the derivatives of the following functions.

(a) $f(x) = 3x^2 - 2x + 10$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{3(x+h)^2 - 2(x+h) + 10 - (3x^2 - 2x + 10)}{h} \\
&= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h + 10 - 3x^2 + 2x - 10)}{h} \\
&= \lim_{h\to 0} \frac{6xh + 3h^2 - 2h)}{h} \\
&= \lim_{h\to 0} 6x + 3h - 2 \\
f'(x) &= 6x - 2
\end{aligned}$$

(b) $g(x) = \sqrt{3x-5}$

Solution

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{\sqrt{3(x+h)-5}-\sqrt{3x-5}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3x+3h-5}-\sqrt{3x-5}}{h} \cdot \frac{\sqrt{3x+3h-5}+\sqrt{3x-5}}{\sqrt{3x+3h-5}+\sqrt{3x-5}} \\
&= \lim_{h\to 0} \frac{(3x+3h-5)-(3x-5)}{h(\sqrt{3x+3h-5}+\sqrt{3x-5})} \\
&= \lim_{h\to 0} \frac{3h}{h(\sqrt{3x+3h-5}+\sqrt{3x-5})} \\
&= \lim_{h\to 0} \frac{3}{(\sqrt{3x+3h-5}+\sqrt{3x-5})} \\
&= \frac{3}{2\sqrt{3x-5}}
\end{aligned}$$

(c) $h(x) = \frac{1}{t^2}$

Solution

$$\begin{aligned}
h'(x) &= \lim_{h\to 0} \frac{\frac{1}{(t+h)^2}-\frac{1}{t^2}}{h} \\
h'(x) &= \lim_{h\to 0} \frac{\frac{1}{(t+h)^2}-\frac{1}{t^2}}{h} \cdot \frac{(t+h)^2t^2}{(t+h)^2t^2} \\
&= \lim_{h\to 0} \frac{t^2-(t+h)^2}{h(t+h)^2t^2} \\
&= \lim_{h\to 0} \frac{-2th+h^2}{h(t+h)^2t^2} \\
&= \lim_{h\to 0} \frac{-2t+h}{(t+h)^2t^2} \\
&= \frac{-2t}{t^4} \\
h'(x) &= \frac{-2}{t^3}
\end{aligned}$$

3. If possible, find the equation of the tangent line to the above curves when $x = 1$.

Solution

(a) $f(1) = 11$, and $f'(1) = 4$. In point-slope form, the tangent line is $y - 11 = 4(x - 1)$, or in slope-intercept form, $y = 4x + 7$.

(b) 1 is not in the domain of g so there is no tangent line.

(c) $h(1) = 1$ and $h'(1) = -2$. In point-slope form, the tangent line is $y - 1 = -2(x - 1)$, or in slope-interept form, $y = -2x + 3$.

4. Find the derivatives of the following functions.

(a) $f(x) = \cos(\sin(x^2))$

Solution

We need to use chain rule twice.

$$f'(x) = -\sin(\sin(x^2)) \cdot \cos(x^2) \cdot 2x$$
$$f'(x) = -2x\sin(\sin(x^2)) \cdot \cos(x^2)$$

(b) $y = \cos(\frac{x}{2})\sqrt{x^2+4}$

Solution

We need chain rule and product rule.

$$\frac{dy}{dx} = -\sin(\frac{x}{2}) \cdot \frac{1}{2} \cdot \sqrt{x^2+4} + \cos\frac{x}{2} \cdot \frac{1}{2}(x^2+4)^{-1/2} \cdot 2x$$
$$= -\frac{1}{2}\sin(\frac{x}{2}) \cdot \sqrt{x^2+4} + x\frac{\cos\frac{x}{2}}{\sqrt{x^2+4}}$$

(c) $\frac{3x^2+1}{x-4}$

Solution

We need quotient rule.

$$\frac{d}{dx}\left(\frac{3x^2+1}{x-4}\right) = \frac{(x-4)(6x) - (3x^2+1)(1)}{(x-4)^2}$$
$$= \frac{3x^2 - 24x - 1}{(x-4)^2}$$

(d) Find $\frac{dy}{dx}$ at $(3,2)$ when $y^2 - 19 = -3xy + x$

Solution

$$\begin{aligned} 2yy' &= -3xy' - 3y + 1 \\ y'(2y+3x) &= -3y + 1 \\ y' &= \frac{3y+1}{2y+3x} \\ y' &= \frac{7}{13} \end{aligned}$$

5. Show that the function $f(x) = x^5 - 2x^3 - 2$ has a zero (root/x-intercept) in the interval $[0,2]$. Make sure to state which theorem you're using and why it applies.

Solution

$f(0) = -2$ and $f(2) = 14$. Since f is continuous on the interval $[0,2]$ and $f(0) < 0$ and $f(2) > 0$ then by the Intermediate value theorem, $f(x) = 0$ for some x in the interval $[0,2]$

6. Show that the function $f(x) = \frac{3x^3-2x^2+1}{x-2}$ is 0 for some real value of x. Make sure to state which theorem you're using and why it applies.

Solution

Note that the question has been altered slightly

x	$f(x)$
-1	$\frac{4}{3}$
0	$-\frac{1}{2}$
1	-2

Since $f(-1) < 0$ and $f(0) > 0$ and $f$is continuous on the interval $(-1,0)$ (it is continuous on its domain, and its domain contains this interval), then by the Intermediate value theorem $f(x) = 0$ for some x in that interval.

7. Consider the following function.

$$f(x) = \begin{cases} 2x+3 & x \leq -2 \\ 1 & -2 < x < 0 \\ 0 & x = 0 \\ -x+4 & x > 0 \end{cases}$$

(a) Graph the function.

(b) Use the graph to determine $\lim_{x \to 0} f(x)$.

Solution

Since $\lim_{x \to 0^-} = 1$ and $\lim_{x \to 0^+} = 4$ then $\lim_{x \to 0}$ does not exist.

(c) Use the graph to determine $\lim_{x \to -2^-} f(x)$.

Solution

$\lim_{x \to -2^-} = -1$

(d) Determine the intervals on which $f(x)$ is continuous. (Be clear about endpoints).

Solution

f is continous on $(-\infty, -2]$, $(-2, 0)$, and $(0, \infty)$.

Note: This is <u>not</u> a union.

(e) Determine the intervals on which $f(x)$ is differentiable. (Be clear about endpoints).

Solution

f is differentiable on $(-\infty, -2)$, $(-2, 0)$, and $(0, \infty)$.