1. Find the equation of the tangent line to the curve at the given point.

   (a) $y = x^2 - 3x$, $(-2, 10)$.

   Solution

$$\begin{aligned} m &= \lim_{h\to 0} \frac{(-2+h)^2 - 3(-2+h) - 10}{h} \\ &= \lim_{h\to 0} \frac{4 - 4h + h^2 + 6 - 3h - 10}{h} \\ &= \lim_{h\to 0} \frac{-7h + h^2}{h} \\ &= \lim_{h\to 0} -7 + h \\ &= -7 \end{aligned}$$

   Therefore the line is of the form $y = -7x + b$. We know $(-2, 10)$ is a point on the line, so $10 = -7(-2) + b$, and thus the equation of the line is

$$y = -7x - 4$$

   (b) $y = \sqrt{5x} + 4$, $(5, 9)$

   Solution

$$\begin{aligned} m &= \lim_{h\to 0} \frac{\sqrt{5(5+h)} + 4 - 9}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{25+5h} - 5}{h} \cdot \frac{\sqrt{25+5h} + 5}{\sqrt{25+5h} + 5} \\ &= \lim_{h\to 0} \frac{(25+5h) - 25}{h(\sqrt{25+5h} + 5)} \\ &= \lim_{h\to 0} \frac{5}{\sqrt{25+5h} + 5} \\ &= \frac{5}{10} = \frac{1}{2} \end{aligned}$$

   Therefore the line is of the form $y = \frac{1}{2}x + b$. We know $(5, 9)$ is a point on the line, so $9 = \frac{1}{2}(5) + b$, and thus the equation of the line is

$$y = \frac{1}{2}x + \frac{13}{2}$$

(c) $x^3 - 2x^2 + 5$, $(1, 4)$.

Solution

$$\begin{aligned} m &= \lim_{h \to 0} \frac{(1+h)^3 - 2(1+h)^2 + 5 - 4}{h} \\ &= \lim_{h \to 0} \frac{-h + h^2 + h^3}{h} \\ &= \lim_{h \to 0} -1 + h + h^2 \\ &= -1 \end{aligned}$$

Therefore the line is of the form $y = -x + b$. We know $(1, 4)$ is a point on the line, so $4 = -1 + b$, and thus the equation of the line is

$$y = -x + 5$$

2. At time $t = 0$ a diver jumps from a platform diving board that is 32 feet above the water. The position of the diver is given by $s(t) = -16t^2 + 16t + 32$ where s is measured in feet and t is measured in seconds. Determine when the diver hits the water, and the divers velocity at impact.

Solution

First, let us determine when the diver hits the water. This will be when he is 0 feet above the water.

$$0 = s(t) = -16(t^2 - t - 2) = -16(t-2)(t+1)$$

So $s(t) = 0$ when $t = -1$ or $t = 2$. Since t represents time, then it must be positive when she reaches the water, so she hits the water at $t = 2$ seconds. To determine the velocity at impact, we need to find $s'(2)$.

$$\begin{aligned} s'(2) &= \lim_{h\to 0} \frac{-16(2+h)^2 + 16(2+h) + 32}{h} \\ &= \lim_{h\to 0} \frac{-16(4 + 4h + h^2 - 2 - h - 2)}{h} \\ &= -16 \lim_{h\to 0} \frac{3h + h^2}{h} \\ &= -16 \lim_{h\to 0} 3 + h \\ &= -16 \cdot 3 = -48 \end{aligned}$$

The diver hits the water after 2 seconds with a velocity of -48 m/s.

3. Suppose $g(6) = 1$ and $g'(6) = -3$. Find the equation of the tangent line to curve $y = g(x)$ when $x = 6$.

Solution

Since $g'(6) = -3$ we know the slope of the tangent line at $x = 6$ is -3. Furthermore, we know that $(6, 1)$ is a point on the line.

$$\begin{aligned} y &= -3x + b \\ 1 &= -3 \cdot 6 + b \\ 19 &= b \end{aligned}$$

The equation of the tangent line is $y = -3x + 19$

4. Suppose the equation of the tangent line to a curve $y = f(x)$ at the point where $x = 2$, is $y = -x + 3$. Find $f(2)$ and $f'(2)$.

Solution

Since the slope of the tangent line is -1, then $f'(2) = -1$.
The curve intersects the tangent line when $x = 2$, so $f(2) = y = -2 + 3 = 1$.

5. Find the derivative of the following functions using the definition of derivative. State the domain of the function and the domain of the derivative.

   (a) $f(x) = \sqrt{x} + 3$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+h} + 3 - (\sqrt{x} + 3)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
f'(x) &= \frac{1}{2\sqrt{x}}
\end{aligned}$$

The domain of $f(x)$ is $[0, \infty)$ and the domain of $f'(x)$ is $(0, \infty)$ (ie. f is differentiable on $(0, \infty)$).

(b) $g(x) = \frac{2+t}{1-2t}$

Solution

$$g'(x) = \lim_{h\to 0} \frac{\frac{2+t+h}{1-2(t+h)} - \frac{2+t}{1-2t}}{h}$$
$$g'(x) = \lim_{h\to 0} \frac{\frac{2+t+h}{1-2t-2h)} - \frac{2+t}{1-2t}}{h} \cdot \frac{(1-2t-2h)(1-2t)}{(1-2t-2h)(1-2t)}$$
$$g'(x) = \lim_{h\to 0} \frac{(2+t+h)(1-2t)-(2+t)(1-2t-2h)}{h(1-2t-2h)(1-2t)}$$
$$g'(x) = \lim_{h\to 0} \frac{5h}{h(1-2t-2h)(1-2t)}$$
$$g'(x) = \lim_{h\to 0} \frac{5}{(1-2t-2h)(1-2t)}$$
$$g'(x) = \frac{5}{(1-2t)^2}$$

The domain of $g(x)$ is $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$ and the domain of $g'(x)$ is $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$.

6. For the following function, find $f'(x)$, $f''(x)$, $f'(3)$, $f''(3)$.

$$f(x) = 3x^2 - 2x + 4$$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{3(x+h)^2 - 2(x+h) + 4 - (3x^2 - 2x + 4)}{h} \\
f'(x) &= \lim_{h\to 0} \frac{6xh + 3h^2 - 2h}{h} \\
f'(x) &= \lim_{h\to 0} 6x + 3h - 2 \\
f'(x) &= 6x - 2
\end{aligned}$$

$$\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{6(x+h) - 2 - (6x - 2)}{h} \\
f''(x) &= \lim_{h\to 0} \frac{6h}{h} \\
f''(x) &= \lim_{h\to 0} 6 \\
f''(x) &= 6
\end{aligned}$$

$$f'(3) = 6(3) - 2 = 16$$

$$f''(3) = 6$$