1. Find the following limits.

(a) $\lim_{x\to 3} \frac{1}{x-3}$

Solution

x	$\frac{1}{x-3}$	x	$\frac{1}{x-3}$
2	-1	4	1
2.5	-2	3.5	2
2.9	-10	3.2	10
2.99	-100	3.01	100
2.999	-1000	3.001	1000

We see that $\lim_{x\to 3^-} \frac{1}{x-3} = -\infty$ and $\lim_{x\to 3^+} \frac{1}{x-3} = \infty$, so $\lim_{x\to 3} \frac{1}{x-3}$ does not exist.

(b) $\lim_{x\to 0} \frac{x^3-4x}{2x+1}$

Solution

Rational functions are continuous on their domain.
The domain of $f(x) = \frac{x^3-4x}{2x+1}$ is $(-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)$, so f is continuous at 0.
This means that $\lim_{x\to 0} f(x) = f(0) = 0$

(c) $\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3}$

Solution

$$\frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \frac{\frac{1}{x}-\frac{1}{3}}{x-3} \cdot \frac{3x}{3x} = \frac{3-x}{3x(x-3} = \frac{-1}{3x}$$

So $\frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \frac{-1}{3x}$ when $x \neq 3$ so $\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} = \lim_{x\to 3} \frac{-1}{3x} = \frac{-1}{9}$
(Note:$\frac{-1}{3x}$ is continuous at $x = 3$)

(d) $\lim_{x\to -2} \frac{1}{x^2-4}$

Solution

x	$\frac{1}{x^2-4}$	x	$\frac{1}{x^2-4}$
-3	.2	-1	$-.3$
-2.5	≈ 0.44	-1.5	≈ -0.57
-2.1	≈ 2.43	-1.9	≈ -2.56
-2.01	≈ 24.9	-1.99	≈ -25
-2.001	≈ 249.9	-1.999	≈ -250

We see that $\lim_{x\to -2^-} \frac{1}{x^2-4} = \infty$ and $\lim_{x\to -2^+} \frac{1}{x^2-4} = -\infty$, so $\lim_{x\to -2} \frac{1}{x^2-4}$ does not exist.

(e) $\lim_{x\to 4} \frac{x^2-7x+12}{2x-8}$

Solution

$\frac{x^2-7x+12}{2x-8} = \frac{(x-3)(x-4)}{2(x-4)} = \frac{x-3}{2}$ when $x \neq 4$, so $\lim_{x\to 4} \frac{x^2-7x+12}{2x-8} = \lim_{x\to 4} \frac{x-3}{2} = \frac{1}{2}$.

2. Find $\lim_{x\to 0^-} x^3 \cos(\frac{2}{x})$. (Hint: Squeeze theorem.)

Solution

First, notice that we have $-1 \leq \cos(\frac{2}{x}) \leq 1$ for all $x \neq 0$. Since we are looking at the left hand limit at 0, consider $x < 0$. Since $x^3 < 0$ we have that

$$-1 \cdot x^3 \geq x^3 \cdot \cos(\frac{2}{x}) \geq 1 \cdot x^3$$

$$-x^3 \geq x^3 \cos(\frac{2}{x}) \geq x^3$$

$$\lim_{x\to 0^-} x^3 = 0 = \lim_{x\to 0^-} -x^3$$

Therefore, by the squeeze theorem, it must be that $\lim_{x\to 0^-} x^3 \cos(\frac{2}{x}) = 0$.

3. Determine where the following function is continuous, and state why.

$$f(x) = \frac{1}{\sqrt{x+3}} - \frac{x^2-1}{x-1} - 4x^2$$

Solution

Let $F(x) = \frac{1}{\sqrt{x+3}}$, $G(x) = \frac{x^2-1}{x-1}$, $H(x) = 4x^2$. $f(x)$ is continuous when F, G, and H are all continuous. Since they are all root functions, polynomials, or rational functions, then they are continuous on their domains.
$F(x)$ is continuous on $(-3, \infty)$
$G(x)$ is continuous on $(\infty, 1) \cup (1, \infty)$
$H(x)$ is coninuous on $(-\infty, \infty)$
Thus, $f(x)$ is continuous on $(-3, 1) \cup (1, \infty)$.

4. Is the following function continuous at $x = 0$? $f(x) = \begin{cases} \frac{x-6}{x-3} & x < 0 \\ 2 & 0 \leq x = 0 \\ \sqrt{4+x^2} & x > 0 \end{cases}$

Solution

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x-6}{x-3} = 2$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sqrt{4+x^2} = 2$$

So $\lim x \to 0 f(x) = 2 = f(0)$ so f is continuous at 0.

5. Use the precise definition of a limit (ϵ and δ) to prove the following limit.

$$\lim_{x\to 10}(3x+5) = 35$$

Solution

Proof. Let $\epsilon > 0$ and $\delta = \epsilon/3$. Then if $|x - 10| < \delta$, and $f(x) = 3x + 5$ we have the following.

$$\begin{aligned} |f(x) - 35| &= |3x + 5 - 35| \\ &= |3x - 30| \\ &= 3|x - 10| \\ &< 3\delta = \epsilon \end{aligned}$$

Thus, for $\epsilon > 0$ $\exists\delta > 0$ such that when $|x - 10| < \delta$,$|f(x) - 35| < \epsilon$, hence $\lim_{x\to 10}(3x + 5) = 35$ □

6. Find where the following function is discontinuous, and state the types of discontinuities.

$$f(x) = \frac{x^2 - 3x + 2}{x^2 - x - 6}$$

Solution

We know that rational functions are continuous on their domains. The domain of $f(x)$ is $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$, so we have discontinuities at $x = -2$ and $x = 3$. By finding the limits as x approaches the discontinuities, we can see that they are both infinite discontinuities.

7. Use the Intermediate Value Theorem to show that $h(x) = 4x^4 - 29x^2 + 25$ has a real root in the inverval $(-3, -2)$.

Solution

$$h(-3) = 88 > 0 \quad h(-2) = -27 < 0$$

By the Intermediate Value Theorem, $h(x) = 0$ for some $x \in (-3, -2)$