Practice Midterm 3
SOLUTIONS
Math 241
Spring 2019

Name:

Read all of the following information before starting the exam:

- **CALCULATORS ARE NOT ALLOWED**.
- Show all work, clearly and in order using proper notations, if you want to get full credit. We reserve the right to take off points if we cannot see how you arrived at your answer (even if your final answer is correct).

1. Use mid-points to approximate the area above the x-axis and under $x^2 + 6$ from $x = 0$ to $x = 6$ using 3 rectangles.

Solution

Each rectangle will have width of $\Delta x = 2$, and we are approximating the heights as the value of f at the midpoints of the intervals $[0, 2], [2, 4], [4, 6]$.

$$\begin{aligned} A &= (1+6)(2) + (9+6)(2) + (25+6)(2) \\ &= 106 \end{aligned}$$

2. Use the definition of area to find the area under $y = x^2$ from $x = 1$ to $x = 3$.

Solution

$$\Delta x = \frac{2}{n} \qquad x_i = 1 + \frac{2i}{n}$$

$$\begin{aligned} \int_1^3 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} (1 + \frac{2i}{n})^2 \cdot \frac{2}{n} \\ &= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} 1 + \frac{4i}{n} + \frac{4i^2}{n^2} \\ &= \lim_{n\to\infty} \frac{2}{n} \left[\sum_{i=1}^{n} 1 + \sum_{i=1}^{n} \frac{4i}{n} + \sum_{i=1}^{n} \frac{4i^2}{n^2} \right] \\ &= \lim_{n\to\infty} \frac{2}{n} \left[n + \frac{4}{n} \sum_{i=1}^{n} i + \frac{4}{n^2} \sum_{i=1}^{n} i^2 \right] \\ &= \lim_{n\to\infty} \frac{2}{n} \left[n + \frac{4}{n} \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ &= \lim_{n\to\infty} 2 + \frac{4n(n+1)}{n^2} + \frac{4n(n+1)(2n+1)}{3n^3} \\ &= 2 + 4 + \frac{8}{3} \\ &= \frac{26}{3} \end{aligned}$$

3. Find the following integrals

   (a) $\int_0^4 2(\sqrt{t} - t) dt$

Solution

$$\begin{aligned}\int_0^4 2(\sqrt{t}-t)dt &= 2\int_0^4 t^{\frac{1}{2}} - tdt \\ &= 2\left[\frac{2t^{\frac{3}{2}}}{3} - \frac{t^2}{2}\right]_0^4 \\ &= 2\left[\frac{16}{3} - 8\right] \\ &= \frac{-16}{3}\end{aligned}$$

(b) $\int \frac{1+2x^3}{x^3}dx$

Solution

$$\begin{aligned}\int \frac{1+2x^3}{x^3}dx &= \int \frac{1}{x^3} + 2dx \\ &= \int x^{-3} + 2dx \\ &= -\frac{x^{-2}}{2} + 2x + C \\ &= -\frac{1}{2x^2} + 2x + C\end{aligned}$$

(c) $\int \tan^4\theta \cdot \sec^2\theta d\theta$

Solution

Let $u = \tan\theta$. Then $du = \sec^2\theta d\theta$.

$$\begin{aligned}\int \tan^4\theta \cdot \sec^2\theta d\theta &= \int u^4 du \\ &= \frac{u^5}{5} + C \\ &= \frac{\tan^5\theta}{5} + C\end{aligned}$$

(d) $\int_0^{\pi} 2\sin x \cos^2 x dx$

Solution

Let $u = \cos x$, so $du = -\sin x dx$. When $x = 0$, $u = 1$, and when $x = \pi$, $u = -1$.

$$\begin{aligned}\int_0^\pi 2\sin x \cos^2 x dx &= \int_1^{-1} 2u^2(-du)\\ &= -\int_1^{-1} 2u^2 du)\\ &= 2\int_{-1}^{1} u^2 du)\\ &= 2\left[\frac{u^3}{3}\right]_{-1}^{1}\\ &= \frac{4}{3}\end{aligned}$$

(e) $\int \frac{x}{(x^2+2)^3} dx$

Solution

Let $u = x^2 + 2$, so $du = 2xdx$ and $\frac{1}{2}du = xdx$.

$$\begin{aligned}\int \frac{x}{(x^2+2)^3} dx &= \int \frac{1}{u^3}\cdot\frac{1}{2}du\\ &= \frac{1}{2}\int u^{-3}du\\ &= \frac{1}{2}\left(\frac{u^{-2}}{-2}\right) + C\\ &= -\frac{1}{4u^2} + C\\ &= -\frac{1}{4(x^2+x)^2} + C\end{aligned}$$

4. Find the area between the curves $y = 2x^2$ and $y = 6x$ from $x = -2$ to $x = 5$.

Solution

The curves intersect at $x = 0$ and $x = 3$. $2x^2 \geq 6x$ on $[-2, 0] \cup [3, 5]$, and $2x^2 \leq 6x$ on $[0, 3]$.

$$\begin{aligned}
A &= \int_{-2}^{0} 2x^2 - 6x dx + \int_{0}^{3} 6x - 2x^2 dx + \int_{3}^{5} 2x^2 - 6x dx \\
&= \left[\frac{2x^3}{3} - 3x^2\right]_{-2}^{0} + \left[3x^2 - \frac{2x^3}{3}\right]_{0}^{3} + \left[\frac{2x^3}{3} - 3x^2\right]_{3}^{5} \\
&= \left[\frac{16}{3} + 12\right] + [27 - 18] + \left[\frac{250}{3} - 75 - 18 + 27\right] \\
&= \frac{131}{3}
\end{aligned}$$

5. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $x = y$, $x \geq 0$ about the line $y = -1$

Solution

The curves intersect at $x = 0, 1, -1$. Since we are only considering $x \geq 0$, then we are looking for

$$V = \int_{0}^{1} A(x) dx$$

where $A(x)$ is the area of a cross-section perpendicular to the x-axis.
Notice each cross section is a washer with inner radois $1 + x^3$ and outer radius $1 + x$, so

$$A(x) = \pi\left((1 + x)^2 - (1 + x^3)^2\right) = \pi(2x + x^2 - 2x^3 - x^6)$$

$$\begin{aligned}
V &= \int_{0}^{1} A(x) dx \\
&= \pi \int_{0}^{1} 2x + x^2 - 2x^3 - x^6 dx \\
&= \pi\left[x^2 + \frac{x^3}{3} - \frac{x^4}{2} - \frac{x^7}{7}\right]_{0}^{1} \\
&= \pi\left[1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{7}\right] \\
&= \frac{29\pi}{42}
\end{aligned}$$