1. Let $f(x) = \cos(x) + 2\sin(x) + x^2$. Use Newtons method to approximate the root in the interval $[-1, 0]$. Let $x_1 = 0$ and find x_4.

Solution

Recall that $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Since $f'(x) = -\sin(x) + 2\cos(x) + 2x$, then

$$x_{n+1} = x_n - \frac{\cos(x) + 2\sin(x) + x^2}{-\sin(x) + 2\cos(x) + 2x}$$

This gives us that

$$x_2 = -0.5$$

$$x_3 = -0.6366699829$$

$$x_4 = -0.6586063411$$

2. Approximate $\sqrt{13}$ correct up to 5 decimal places.

Solution

To approximate $\sqrt{13}$ we want to find the positive root of $x^2 - 13 = 0$. Let us make an initial guess of $x_1 = 3$. Then we have the following:

$$x_2 = \frac{10}{3} \equiv 3.666666666$$
$$x_3 \equiv 3.606060606$$
$$x_4 \equiv 3.605551312$$
$$x_5 \equiv 3.605551275$$

So up to 5 decimal places we have that $\sqrt{13} = 3.60555$.

3. Consider the function $f(x) = x^2 - 3x + 1$. Let x_1 be $1, 2, 3, 4$. What is x_2 in each situation? Which is the best first approximation?

Solution

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

x_1	x_2
1	0
2	3
3	$\frac{8}{3}$
4	3

1 and 3 are the best first approximations. They are approximating different roots.

4. Find all anti-derivatives of the following functions.

(a) $f(t) = \frac{1}{\sqrt[3]{t}}$

Solution

$f(t) = t^{-1/3}$ so if $F'(t) = f(t)$ then $F(t) = \frac{t^{2/3}}{\frac{2}{3}} + C = \frac{3t^{2/3}}{2} + C$ for some real number C.

(b) $f(x) = \pi\cos(x) + x^5$

Solution

If $F'(t) = f(t)$ then $F(t) = \pi\sin(x) + \frac{x^6}{6} + C$ for some real number C.

(c) $f(x) = \sec^2(x) + \sec(x)\tan(x)$

Solution

If $F'(t) = f(t)$ then $F(t) = \tan(x) + \sec(x) + C$ for some real number C.

(d) $g(x) = \frac{2x^3 - \sqrt{x}}{2x}$

Solution

$g(x) = \frac{2x^3}{2x} - \frac{\sqrt{x}}{2x} = x^2 - \frac{1}{2\sqrt{x}}$. If $G'(t) = g(t)$ then $G(t) = \frac{x^3}{3} - \sqrt{x} + C$.

5. Find $f(x)$ when $f''(x) = 12x - 8$, $f'(1) = 4$, and $f(1) = 3$.

Solution

$$f'(x) = 12 \cdot \frac{x^2}{2} - 8x + C = 6x^2 + 8x + C_1$$

Since $f'(1) = 4$, then $C_1 = 6$, so

$$f'(x) = 6x^2 - 8x + 6$$

$$f(x) = 6 \cdot \frac{x^3}{3} - 8 \cdot \frac{x^2}{2} + 6x + C_2 = 2x^3 - 4x^2 + 6x + C_2$$

Since $f(1) = 3$, then $C_2 = -1$, so

$$f(x) = 2x^3 - 4x^2 + 6x - 1$$

6. Given that the graph of f passes through the point $(1, 6)$ and that the slope of its tangent line at $(x, f(x)$ is $2 - 3x$, find $f(1)$.

Solution

Since the slope of the tangent line at a point is $2 - 3x$ then $f'(x) = 2 - 3x$. This implies that $f(x) = 2x - \frac{3}{2}x^2 + C$. Since $f(1) = 6$, then $c = \frac{-1}{2}$, so $f(x) = 2x - \frac{3}{2}x^2 - \frac{1}{2}$

7. Find a function f such that $f'(x) = 3x^2$ and the line $3x - y = 4$ is tangent to the graph of f.

Solution

The slope of the line $3x - y = 4$ is 3. $f'(x) = 3$ when $x = \pm 1$.
If $x = 1$, then the tangent line intersects the curve at $(1, -1)$, so $f(1) = -1$. The antiderivative of $f'(x)$ is $f(x) = x^3 + c$, so c must be -2.

$$f(x) = x^3 - 2$$

If $x = -1$, then the tangent line intersects the curve at $(-1, -7)$, so $f(-1) = -7$. The antiderivative of $f'(x)$ is $f(x) = x^3 + c$, so c must be -6.

$$f(x) = x^3 - 6$$