1. Consider the function $f(x) = x^4 - x^2$.

- Find the domain of $f$.

**Solution**

$f$ is a polynomial so the domain is $(-\infty, \infty)$.

- Find the $x$ and $y$ intercepts of the graph of $f(x)$.

**Solution**

$x$-int: $0 = x^4 - x^2 = x^2(x+1)(x-1)$, so $x = 0, -1, 1$
$y$-int: $f(0) = 0$ so $y = 0$

- Determine symmetry.

**Solution**

$f(x) = f(-x)$ so $f$ is even, and thus symmetric with respect to the $y$-axis.

- Find all asymptotes.

**Solution**

None

- Determine when $f$ is increasing and decreasing.

**Solution**

$$f'(x) = 4x^3 - 2x$$
$$0 = 2x(2x^2 - 1)$$
$$x = 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}.$$

We can test values outside and in between these values to determine the intervals in which $f$ is increasing or decreasing. Decreasing: $(-\infty, -\frac{1}{\sqrt{2}}) \cup (0, \frac{1}{\sqrt{2}})$
Increasing: $(-\frac{1}{\sqrt{2}}, 0) \cup (\frac{1}{\sqrt{2}}, \infty)$

- Determine the concavity of $f$.

**Solution**

$$f''(x) = 12x^2 - 2$$
$$0 = 12x^2 - 2$$
$$\pm\frac{1}{\sqrt{6}} = x$$

Concave up: $(-\infty, -\frac{1}{\sqrt{6}}) \cup (\frac{1}{\sqrt{6}}, \infty)$
Concave down: $(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$

- Sketch the curve $y = f(x)$.

2. Consider the function $f(x) = \frac{x}{\sqrt{x^2+1}}$

- Find the domain of $f$.

**Solution**

$(-\infty, \infty)$

- Find the $x$ and $y$ intercepts of the graph of $f(x)$.

**Solution**

$x$-int: $x = 0$
$y$-int: $f(0) = 0$, so $y = 0$

- Determine symmetry.

**Solution**

$f(-x) = -f(x)$ so $f$ is odd, and thus symmetric with respect to the origin.

- Find all asymptotes.

**Solution**

Since the denominator is never 0, there are no vertical asymptotes. For the horizontal asymptote, we will take the limit as $x$ approaches $\pm\infty$. Recall that if $x$ is positive, $x = \sqrt{x^2}$, and if $x$ is negative, $x = -\sqrt{x^2}$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{1}{\frac{\sqrt{x^2+1}}{x}} \\
&= \lim_{x\to\infty} \frac{1}{\frac{\sqrt{x^2+1}}{\sqrt{x^2}}} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{x^2+1}{x^2}}} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} \\
&= \frac{1}{\sqrt{\lim_{x\to\infty} 1+\frac{1}{x^2}}} \\
&= \frac{1}{\sqrt{1}} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to-\infty} \frac{1}{\frac{\sqrt{x^2+1}}{x}} \\
&= \lim_{x\to-\infty} \frac{1}{\frac{\sqrt{x^2+1}}{-\sqrt{x^2}}} \\
&= \lim_{x\to-\infty} \frac{1}{-\sqrt{\frac{x^2+1}{x^2}}} \\
&= \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{1}{x^2}}} \\
&= \frac{1}{-\sqrt{\lim_{x\to\infty} 1+\frac{1}{x^2}}} \\
&= -1
\end{aligned}$$

So there are horizontal asymptotes at $y = \pm 1$.

- Determine when $f$ is increasing and decreasing.

**Solution**

$$
\begin{aligned}
f'(x) &= \frac{\sqrt{x^2+1} - 2x^2\left(\frac{1}{2\sqrt{x^2+1}}\right)}{x^2+1} \\
&= \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} \\
&= \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1} \cdot \frac{\sqrt{x^2+1}}{\sqrt{x^1+1}} \\
&= \frac{x^2+1-x^2}{(x^2+1)^{3/2}} \\
&= \frac{1}{(x^2+1)^{3/2}}
\end{aligned}
$$

$f'(x)$ is never 0 and is always positive, so $f$ is always increasing.

- Determine the concavity of $f$.

**Solution**

$$
\begin{aligned}
f''(x) &= -\frac{3}{2}(x^2+1)^{-5/2}(2x) \\
&= \frac{-3x}{(x^2+1)^{5/2}}
\end{aligned}
$$

So $f''(x) = 0$ when $x = 0$. We can test positive and negative values to determine concavity. Concave up: $(-\infty, 0)$
Concave down: $(0, \infty)$

- Sketch the curve $y = f(x)$.

3. The curve of $f(x) = \frac{x^3+2x^2-4}{x^2+2}$ has a slant asymptote. Find the slant asymptote.

**Solution**

Doing polynomial division we can see that

$$f(x) = x + 2 - \frac{2x+8}{x^2+2}$$

, so there is a slant asymptote at $x + 2$ since as $x$ approaches $\pm\infty$, $-\frac{2x+8}{x^2+2}$ approaches 0.

4. The sum of two positive numbers is 16. Find the smallest possible value of the sum of their squares.

**Solution**

Let $a$ and $b$ be the two numbers such that $a + b = 16$. If $S = a^2 + b^2$, then we want to find the minimum value of $S$. Let us substitute $b = 16 - a$, and find the critical numbers of $S$.

$$\begin{aligned} S(a) &= a^2 + (16-a)^2 \\ S(a) &= 2a^2 - 32a + 256 \\ S'(a) &= 4a - 32 \\ 0 &= 4a - 32 \\ a &= 8 \end{aligned}$$

Since $S'(a) < 0$ when $a < 8$ and $S'(a) > 0$ when $a > 8$ then the absolute minimum value of $S$ must be when $a = 8$. Since $S'(8) = 32$, the minimum possible sum of the squares is 32.

5. A rectangle is bounded by the x- and y-axes and the graph of $y = 5 - \frac{1}{2}x$. What length and width should the rectangle have so that its area is a maximum?

**Solution**

If the length of such a rectangle is $x$, then the width is $y$ where $y = 5 - \frac{1}{2}x$. Thus the area of the rectangle is given by the equation $A(x) = x(5 - \frac{1}{2}x)$. We should find the critical numbers to determine any absolute maximums.

$$A'(x) = 5 - x$$
$$0 = 5 - x$$
$$x = 5$$

Since $A'(x)$ is positive for $x < 5$ and negative for $x > 5$ then there must be an absolute maximum when $x = 5$. The length of rectangle at this maximum area is 5 and the width must be $\frac{5}{2}$.

6. An open box with a rectangular base is to be constructed from a 16 in. by 21 in. piece of cardboard by cutting out squares from each corner and bending up the sides. Find the dimensions of the box that will have the largest volume.

**Solution**

Let $x$ represent the side length of the square cut out at each corner. Then the height of the box is $x$, the width is $21 - 2x$ and the length is $16 - 2x$. Notice that $0 < x < 8$, otherwise it wouldnt be a box. The volume is thus representated by the equation $V(x) = x(21 - 2x)(16 - 2x)$. We want to find the critical numbers to determine the absolute maximum value.

$$V'(x) = 12x^2 - 148x + 336 = 4(x - 3)(3x - 28)$$

So we can see the critical numbers are at $x = 3$ and $x = \frac{28}{3}$, however, since $x < 8$, then we only need to consider $x = 3$. Since $V'(1) > 0$ and $V'(4) < 0$ then there must be a absolute maximum of $V$ at $x = 3$. Thus, the dimensions of the box that has the largest volume is 10 in by 15 in by 3 in.